

SPRINGER CORRESPONDENCES FOR DIHEDRAL GROUPS

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ABSTRACT. Recent work by a number of people has shown that complex reflection groups give rise to many representation-theoretic structures (*e.g.*, generic degrees and families of characters), as though they were Weyl groups of algebraic groups. Conjecturally, these structures are actually describing the representation theory of as-yet undescribed objects called *spetses*, of which reductive algebraic groups ought to be a special case.

In this paper, we carry out the Lusztig–Shoji algorithm for calculating Green functions for the dihedral groups. With a suitable set-up, the output of this algorithm turns out to satisfy all the integrality and positivity conditions that hold in the Weyl group case, so we may think of it as describing the geometry of the “unipotent variety” associated to a spets. From this, we determine the possible “Springer correspondences”, and we show that, as is true for algebraic groups, each special piece is rationally smooth, as is the full unipotent variety.

1. INTRODUCTION

Many constructions arising in the representation theory of reductive algebraic groups really depend only on the Weyl group. In recent years, it has been discovered that many of these constructions can be generalized to the setting of complex reflection groups (*e.g.*, cyclotomic Hecke algebras [2, 3, 9], generic degrees [22], root lattices (on \mathbb{Z} [28], on a ring of algebraic integers [27]), families of characters [29, 7, 15, 25]). Indeed, it has been conjectured that these constructions actually describe the representation theory of some as-yet undescribed algebraic object called a *spets*. In this paper, we add to this list by studying the “geometry of the unipotent variety” associated to the dihedral groups, via the Lusztig–Shoji algorithm for computing Green functions (see [30], [18, §24]).

It has already been observed by various people that this algorithm is something that lends itself to generalization to complex reflection groups (see [13, 31, 32]). We will review the algorithm in detail later, but for now, let us simply recall that in order to carry out the algorithm for a Weyl group W of an algebraic group, one must first have some information about the Springer correspondence for that group. At a rudimentary level, the required information is the partitioning of $\text{Irr}(W)$ into disjoint subsets, one for each unipotent class.

If we are working with a complex reflection group W , how do we choose such a partition? In [13], Geck and Malle used families of characters as the subsets of the partition. Of course, when W is a Weyl group, this is quite different from the “correct” partitioning by the Springer correspondence. Nevertheless, it seems likely to have geometric meaning: they conjectured that the algorithm in this form can be used to compute the number of \mathbb{F}_q -points in a special piece of the unipotent variety

(see [13, Conjectures 2.5–2.7], as well as [20, 33]). Separately, Shoji [31, 32] has studied the algorithm for imprimitive complex reflection groups by partitioning the characters using combinatorial objects called “symbols” (these are generalizations of the symbols and u -symbols that occur in the representation theory of algebraic groups of classical type).

An important feature of the original Lusztig–Shoji algorithm is that its output obeys certain integrality and positivity conditions. This is a consequence of geometric considerations on the unipotent variety; the algorithm itself, *a priori*, need not obey them. However, Geck and Malle conjecture that their version of the algorithm also satisfies these properties, and Shoji proves that some of them hold for his version as well.

In this paper, we take a somewhat different approach: rather than fixing a partition in advance, we consider all partitions subject to certain initial constraints, and we seek to identify those (if any) for which the output of the algorithm satisfies appropriate integrality and positivity conditions. For W a finite complex reflection group and χ an irreducible character of it, let b_χ denote the largest power of q dividing the fake degree of χ . Then we choose a cyclotomic Hecke algebra \mathcal{H} for W and we fix a set \mathcal{S} of irreducible characters of W including all *special* characters. (Here χ is said to be special if b_χ is equal to the largest power of q dividing the generic degree of χ , where the latter is defined with respect to the canonical symmetrizing trace on \mathcal{H} .) We look for partitions of $\text{Irr}(W)$ into disjoint subsets \mathcal{C} such that (among other conditions) each \mathcal{C} contains a unique member (called the *Springer character* of \mathcal{C}) on which b attains its minimal value and the set of all Springer characters is precisely \mathcal{S} . In the case when W is the Weyl group of a reductive connected algebraic group G then we can take for the Springer characters of W all the irreducible characters which correspond, via the (actual) Springer correspondence, with the trivial local system on a unipotent class. In particular, this set of characters of W is in bijection with the set of unipotent classes in G and it depends on G itself.

Remarkably, when W is a dihedral group, it turns out that in most cases (in all cases, under a minor additional condition), for a given set \mathcal{S} , there is a *unique* partition of the desired sort. In view of this uniqueness, it seems justified to refer to that partition as “the” Springer correspondence with respect to \mathcal{S} . (Moreover, our results are compatible with the various “true” Springer correspondences when the dihedral group is a Weyl group of type A_2 , B_2 , or G_2 —see section 4.) For that partition, we can then interpret the output of the Lusztig–Shoji algorithm as giving information about the geometry of a hypothetical “unipotent variety” for the dihedral group. In particular, we find that, as is true for algebraic groups, the unipotent variety is rationally smooth [5], as is each special piece [16, 20].

We make all this precise in Section 2, where we review the Lusztig–Shoji algorithm, and define the various conditions and constraints mentioned above. In Section 3, we review some basic facts about the dihedral group, and we give precise statements (and proofs for some) of the main results. This section also includes a tabulation of the output of the Lusztig–Shoji algorithm. The one remaining task is the proof of the uniqueness result mentioned above; this occupies the entirety of Section 5.

For Weyl groups, the Springer characters are precisely those which arise as $j_{W'}^W \chi$, where j denotes truncated induction, W' is the stabilizer in W of a point of the

maximal torus, and χ is a special character of W' . In [1], the authors have studied an analogue of this procedure in the case of spetsial complex reflection groups, including the dihedral groups. In this way, one obtains a preferred set \mathcal{S}_{pf} of Springer characters. The description of \mathcal{S}_{pf} for the dihedral groups will be recalled in Section 6.

On the other hand, it is very natural to consider the dihedral groups as a class of non-crystallographic Coxeter groups. Then Krilloff and Ram have proved in [17, §3.4] that, as for Weyl groups, there exists a one-to-one correspondence between the irreducible characters of a dihedral group W and the tempered simple \mathbb{H} -modules with real central characters, where \mathbb{H} is a graded Hecke algebra associated to W . Such a correspondence, combined with the partition of irreducible characters of W defined here, will provide a partition of the set of tempered simple \mathbb{H} -modules with real central characters.

2. THE LUSZTIG-SHOJI ALGORITHM

Let W be a finite complex reflection group acting on a vector space V . Let q be an indeterminate and $P_W(q)$ denote the Poincaré polynomial of W :

$$P_W(q) = \prod_{i=1}^{\dim V} \frac{q^{d_i} - 1}{q - 1},$$

where $d_1, \dots, d_{\dim V}$ are the degrees of W . Next, for any class function f of W , we define a polynomial $R(f)$ by

$$R(f)(q) = \frac{(q-1)^{\dim V} P_W(q)}{|W|} \sum_{w \in W} \frac{\det_V(w) f(w)}{\det_V(q \cdot \text{id}_V - w)}.$$

If χ is an irreducible character of W , then $R(\chi)$ is commonly known as the *fake degree* of χ . In this case, we define b_χ to be the largest power of q dividing $R(\chi)$. We note that (see [23, (6.2)])

$$(1) \quad R(\overline{\det_V \otimes f})(q) = q^{N^*} R(f)(q^{-1}).$$

(Here N^* is the number of reflections in W , and $\overline{}$ is the complex conjugation. Of course, for a Coxeter group, \det_V and $\overline{\det_V}$ are both just the sign character. Hence the equation (1) is a generalization to complex reflection groups of [10, Proposition 11.1.2].)

Next, let Ω be the square matrix with rows and columns indexed by $\text{Irr}(W)$ defined as follows:

$$\Omega = (\omega_{\chi, \chi'})_{\chi, \chi' \in \text{Irr}(W)}, \quad \omega_{\chi, \chi'} = q^{N^*} R(\chi \otimes \chi' \overline{\det_V}).$$

Definition 2.1. A *Lusztig-Shoji datum* for a complex reflection group W is a triple $(X, <, a)$, where $X = \{\mathcal{C}\}$ is a partition of $\text{Irr}(W)$ into disjoint subsets (that is, $\text{Irr}(W) = \bigsqcup_{\mathcal{C} \in X} \mathcal{C}$); the relation $<$ is a total order on X ; and $a : X \rightarrow \mathbb{N}$ is an order reversing function. The member of X to which a given $\chi \in \text{Irr}(W)$ belongs is called its *support*, and the statement $\text{supp } \chi = \mathcal{C}$ is equivalent to the statement that $\chi \in \mathcal{C}$.

Definition 2.2. A *system of Green functions* with respect to a Lusztig-Shoji datum $(X, <, a)$ is a solution to the matrix equation

$$(2) \quad PAP^t = \Omega,$$

where P and Λ are also square matrices of rational functions over $\mathbb{Z}[q]$ with rows and columns indexed by $\text{Irr}(W)$, subject to the following conditions:

$$(3) \quad \begin{aligned} P_{\chi, \chi'} &= \begin{cases} 0 & \text{if } \text{supp } \chi < \text{supp } \chi', \\ \delta_{\chi, \chi'} q^{a_c} & \text{if } \text{supp } \chi = \text{supp } \chi' = \mathcal{C}, \end{cases} \\ \Lambda_{\chi, \chi'} &= 0 \quad \text{if } \text{supp } \chi \neq \text{supp } \chi'. \end{aligned}$$

The following notation for picking out certain submatrices of these matrices will be useful:

$$P_{\chi, \mathcal{C}} = (P_{\chi, \chi'})_{\chi' \in \mathcal{C}} \quad \Lambda_{\mathcal{C}} = (\Lambda_{\chi, \chi'})_{\chi, \chi' \in \mathcal{C}} \quad \Omega_{\mathcal{C}, \mathcal{C}'} = (\omega_{\chi, \chi'})_{\chi \in \mathcal{C}, \chi' \in \mathcal{C}'}$$

We have the following fundamental fact:

Proposition 2.3 (Lusztig, Shoji, Geck–Malle). *Every Lusztig–Shoji datum admits a unique system of Green functions.*

For a proof, see [13, Proposition 2.2]. (In *loc. cit.*, the proposition is stated only for finite Coxeter groups, and only for a certain specific Lusztig–Shoji datum, but the proof is in fact completely general.) In the course of the proof, one obtains an inductive formula for computing P and Λ , as follows: Given $\mathcal{C} \in X$, suppose that the blocks $P_{\chi, \mathcal{C}'}$ and $\Lambda_{\mathcal{C}'}$ are known for all $\mathcal{C}' < \mathcal{C}$ and all $\chi \in \text{Irr}(W)$. Then $\Lambda_{\mathcal{C}}$ and $P_{\chi, \mathcal{C}}$ are given by

$$(4) \quad \Lambda_{\mathcal{C}} = q^{-2a_{\mathcal{C}}} \left(\Omega_{\mathcal{C}, \mathcal{C}} - \sum_{\mathcal{C}' < \mathcal{C}} P_{\mathcal{C}, \mathcal{C}'} \Lambda_{\mathcal{C}'} P_{\mathcal{C}, \mathcal{C}'}^t \right)$$

$$(5) \quad P_{\chi, \mathcal{C}} = q^{-a_{\mathcal{C}}} \left(\Omega_{\chi, \mathcal{C}} - \sum_{\mathcal{C}' < \mathcal{C}} P_{\chi, \mathcal{C}'} \Lambda_{\mathcal{C}'} P_{\mathcal{C}, \mathcal{C}'}^t \right) \Lambda_{\mathcal{C}}^{-1} \quad \text{if } \text{supp } \chi > \mathcal{C}.$$

(Of course, if $\text{supp } \chi \leq \mathcal{C}$, then $P_{\chi, \mathcal{C}}$ is determined by (3).)

We choose a cyclotomic Hecke algebra \mathcal{H} for W (see [9]). Then \mathcal{H} admits a canonical symmetrizing trace. Indeed, the form defined in [6] satisfies (1)(a) and (1)(b) of [8, Theorem-Assumption 2] (the fact that it turns \mathcal{H} into a symmetric algebra is proved in [24]); on the other hand, since it follows from [11] that the corresponding Schur elements are as conjectured in [22, Vermutung 1.18], the form satisfies also (1)(c) of [8, Theorem-Assumption 2] by [8, Lemma 2.7]. Such a form is unique.

Then one can consider the generic degrees of the characters of W defined with respect to \mathcal{H} and the above symmetrizing trace. Recall that a character $\chi \in \text{Irr}(W)$ is said to be *special* if b_{χ} is equal to the largest power of q dividing the generic degree of χ . In addition, \mathcal{H} determines the families of characters of $\text{Irr}(W)$ (see [29], [25]), which play a role below.

In the following definition, we list a number of desirable properties that a Lusztig–Shoji datum may have.

Definition 2.4. Choose a cyclotomic Hecke algebra \mathcal{H} for W . A Lusztig–Shoji datum $(X, <, a)$ is said to be a *Springer correspondence* for W and \mathcal{H} if the following additional conditions are satisfied:

- (1) For each \mathcal{C} , we have $a_{\mathcal{C}} = \min\{b_{\chi} \mid \chi \in \mathcal{C}\}$. Moreover, there is a unique member of \mathcal{C} (called the *Springer representation* of \mathcal{C}) on which b attains its minimum value.

- (2) Every special representation of W occurs as a Springer representation of some $\mathcal{C} \in X$.
- (3) If χ' is a nonspecial representation in the same family as the special representation χ , then $\text{supp } \chi' \leq \text{supp } \chi$.
- (4) The entries of Λ are polynomials with integer coefficients, and the entries of P are polynomials with nonnegative integer coefficients.
- (5) If $\chi \in \mathcal{C}$, then $P_{\chi, \chi'}$ is divisible by $q^{a_{\mathcal{C}}}$ for all χ' .

In this case, the sets \mathcal{C} are called *unipotent classes*. A class is *special* if its Springer representation is.

Remark 2.5. Let W be the Weyl group of a reductive connected algebraic group G . We assume that G is defined over \mathbb{F}_q with Frobenius map F and that there exists a maximal torus T of G which is defined and split over \mathbb{F}_q . Recall that the (actual) Springer correspondence is a map ν from $\text{Irr}(W)$ to the set of pairs (C, \mathcal{E}) , where C is a unipotent G -conjugacy class in G and \mathcal{E} is a G -equivariant irreducible local system on C . Let X be the partition of $\text{Irr}(W)$ induced by ν , that is, $X = \{\mathcal{C}\}$, where $\mathcal{C} = \mathcal{C}_C$ is the set of $\chi \in \text{Irr}(W)$ such that $\nu(\chi) = (C, \mathcal{E})$ for some G -equivariant irreducible local system \mathcal{E} on C . Let $a: X \rightarrow \mathbb{N}$ be the function defined by $a_{\mathcal{C}} := \dim \mathcal{B}_u$, where \mathcal{B}_u is the variety of Borel subgroups of G containing $u \in C$.

All the above properties (1)–(5) hold in this case:

- (1) If $\chi \in \mathcal{C}$ then we have $a_{\mathcal{C}} = \dim \mathcal{B}_u \leq b_{\chi}$ (see [35, §1.1]). Moreover, if $\nu(\chi) = (C, \mathbb{Q}_{\ell})$ then we have $a_{\mathcal{C}} = b_{\chi}$ (see [4, Corollary 4]). Hence (1) is satisfied and the “Springer representations” of W are the irreducible representations of W which correspond, via the (actual) Springer correspondence, with the trivial local system on a unipotent class. In particular, the set of Springer representations is in bijection with the set of unipotent classes.
- (2) For $\chi \in \text{Irr}(W)$ let a'_{χ} denote the largest power of q dividing the generic degree of χ . We have $a'_{\chi} \leq a_{\mathcal{C}}$ if $\chi \in \mathcal{C}$ (see [19, Cor. 10.9]). In particular, when χ is special, it then follows from (1) that $a'_{\chi} = a_{\mathcal{C}} = b_{\chi}$.
- (3) Property (3) is satisfied (see [12, Proposition 2.2]).
- (4) Let $\tilde{\Omega} = (\tilde{\omega}_{\chi, \chi'})_{\chi, \chi' \in \text{Irr}(W)}$, where

$$\tilde{\omega}_{\chi, \chi'} = q^{\dim T - 1/2(\text{codim } C + \text{codim } C')} \cdot \frac{|G^F|}{|W|} \cdot \sum_{w \in W} \chi(w) \chi'(w) \cdot |T_w^F|^{-1},$$

where $\nu(\chi) = (C, \mathcal{E})$ and $\nu(\chi') = (C', \mathcal{E}')$.

Using the fact that $a_{\mathcal{C}} = 1/2(\text{codim } C - \dim T)$ (see [10, (5.10.1)]), we see that

$$\omega_{\chi, \chi'} = q^{a_{\mathcal{C}} + a_{\mathcal{C}'}} \cdot \tilde{\omega}_{\chi, \chi'}.$$

Lusztig proved in [18, §24] (see especially [18, (24.5.2)]) that the equation $\tilde{P} \Lambda \tilde{P}^t = \tilde{\Omega}$ with respect to the unknown variables Λ , \tilde{P} admits a solution with $\Lambda_{\chi, \chi'} \in \mathbb{Z}[q]$, $\tilde{P}_{\chi, \chi'} \in \mathbb{Z}[q]$ and

$$\begin{aligned} \tilde{P}_{\chi, \chi'} &= \begin{cases} 0 & \text{if } \text{supp } \chi < \text{supp } \chi', \\ 1 & \text{if } \text{supp } \chi = \text{supp } \chi' = \mathcal{C}, \end{cases} \\ \Lambda_{\chi, \chi'} &= 0 \quad \text{if } \text{supp } \chi \neq \text{supp } \chi'. \end{aligned}$$

Then the matrices P and Λ with $P_{\chi, \chi'} = q^{ac} \tilde{P}_{\chi, \chi'}$ where $\nu(\chi) = (C, \mathcal{E})$ satisfy the condition (3) in Definition 2.2.

(5) Since $\tilde{P}_{\chi, \chi'} \in \mathbb{Z}[q]$, we have that $P_{\chi, \chi'}$ is divisible by q^{ac} for all $\chi' \in \text{Irr}(W)$.

Any system of Green functions gives rise to a partial order \preceq on X that is compatible with, but in general weaker than, the order $<$, as follows: \preceq is the transitive closure of the relation

$$\mathcal{C} \preceq \mathcal{C}' \quad \text{if there exist } \chi \in \mathcal{C} \text{ and } \chi' \in \mathcal{C}' \text{ such that } P_{\chi, \chi'} \neq 0.$$

In the case of a Springer correspondence, we call this the *closure order* on unipotent classes. A *special piece* is then defined just as for algebraic groups: it is the union of a special class and all those nonspecial classes in its closure that are not also in the closure of any smaller special class.

3. THE DIHEDRAL GROUPS

Our work on the dihedral groups will take place in the following framework: we fix a set $\mathcal{S} \subset \text{Irr}(W)$, including all special characters, and we look for Springer correspondences $(X, <, a)$ whose set of Springer representations is precisely \mathcal{S} .

Remark 3.1. It seems reasonable to fix the set of Springer representations in advance because for Weyl groups, there is an elementary way to compute the set of Springer representations without knowing the full Springer correspondence. (The set of Springer representations is precisely the set of representations arising as $j_{W'}^W \chi$, where j denotes truncated induction, W' is the stabilizer in W of a point of the maximal torus, and χ is a special character of W' , see [10, § 12.6].)

An analogue of this procedure in the dihedral groups provides a preferred set \mathcal{S}_{pf} of Springer representations, to which one can apply the results obtained here (see section 6).

Henceforth, we work only with the dihedral group $W = \text{I}_2(m)$. We begin by recalling some facts about the representation theory of $\text{I}_2(m)$. Its irreducible representations are:

$$\chi_0, \chi_1, \dots, \chi_{\lfloor \frac{m-1}{2} \rfloor}, \epsilon; \quad \text{and} \quad \chi_r, \chi'_r \quad \text{if } m = 2r.$$

Here, χ_0 is the trivial representation, ϵ is the sign representation, and we have

$$b_{\chi_i} = i \text{ for all } i, \quad b_{\chi'_r} = r, \quad \text{and} \quad b_{\epsilon} = m.$$

The representations $\chi_1, \dots, \chi_{\lfloor (m-1)/2 \rfloor}$ are all 2-dimensional, while χ_0 , ϵ , and χ_r and χ'_r are 1-dimensional. The special representations are χ_0 , χ_1 , and ϵ .

The matrix Ω is described in the following table. Recall that Ω is symmetric; below, to reduce clutter, we have only recorded the part of Ω below the diagonal. In the table below i and j are assumed to be non-zero.

Ω	χ_0	χ_i	χ_r	χ'_r	ϵ
χ_0	q^{2m}				
χ_j	$q^{m+j} + q^{2m-j}$	$q^{m+ i-j } + q^{m+i+j} + q^{2m-i-j} + q^{2m- i-j }$			
χ_r	$q^{\frac{3}{2}m}$	$q^{\frac{3}{2}m-i} + q^{\frac{3}{2}m+i}$	q^{2m}		
χ'_r	$q^{\frac{3}{2}m}$	$q^{\frac{3}{2}m-i} + q^{\frac{3}{2}m+i}$	q^m	q^{2m}	
ϵ	q^m	$q^{m+i} + q^{2m-i}$	$q^{\frac{3}{2}m}$	$q^{\frac{3}{2}m}$	q^{2m}

Let \mathcal{S} be a set of irreducible representations of $\text{I}_2(m)$ including χ_0 , χ_1 , and ϵ . In case m is even and \mathcal{S} contains exactly one of χ_r and χ'_r , we assume henceforth,

without loss of generality, that it in fact contains χ'_r . (This assumption will allow us to simplify some formulas by treating χ_r and the various χ_i with $i < r$ uniformly.)

Let us define a sequence of integers

$$(6) \quad d_0 < d_1 < \cdots < d_N < m/2 \quad \text{where} \quad d_0 = 0 \text{ and } d_1 = 1$$

by $\{\chi_i \in \mathcal{S} \mid i < m/2\} = \{\chi_{d_0}, \dots, \chi_{d_N}\}$. We will show that $I_2(m)$ admits a Springer correspondence whose set of Springer representations is precisely \mathcal{S} . We will use the following notation for unipotent classes:

$$\begin{aligned} \mathcal{C}_k &: && \text{class with Springer representation } \chi_{d_k} \\ \mathcal{C}_{\chi_r}, \mathcal{C}_{\chi'_r}, \mathcal{C}_\epsilon &: && \text{classes with Springer representations } \chi_r, \chi'_r, \epsilon, \text{ respectively} \end{aligned}$$

Note that \mathcal{C}_ϵ is automatically a singleton, since there are no characters χ with $b_\chi > b_\epsilon$. Similarly, if $\chi'_r \in \mathcal{S}$, then $\mathcal{C}_{\chi'_r}$ must be a singleton: the only character χ with $b_\chi > b_{\chi'_r}$ is ϵ , which is already the Springer representation of another class. The same argument applies to \mathcal{C}_{χ_r} if $\chi_r \in \mathcal{S}$.

Theorem 3.2. *Let \mathcal{S} and d_0, \dots, d_N be as above. $I_2(m)$ admits a Springer correspondence whose set of Springer representations is precisely \mathcal{S} . Every such Springer correspondence has the form*

$$\begin{aligned} \mathcal{C}_0 &= \{\chi_0\} \\ \mathcal{C}_k &= \{\chi_{d_k}, \chi_{d_k+1}, \dots, \chi_{d_{k+1}-1}\} \cup \{\chi_{f_{k+1}+1}, \chi_{f_{k+1}+2}, \dots, \chi_{f_k}\} \text{ for } 1 \leq k \leq N-1 \\ \mathcal{C}_N &= \begin{cases} \{\chi_{d_N}, \chi_{d_N+1}, \dots, \chi_{\lfloor (m-1)/2 \rfloor}\} & \text{if } m \text{ odd, or } m \text{ even and } \chi_r, \chi'_r \in \mathcal{S} \\ \{\chi_{d_N}, \chi_{d_N+1}, \dots, \chi_{f_N}\} & \text{if } m \text{ even, } \chi_r \notin \mathcal{S} \text{ and } \chi'_r \in \mathcal{S} \\ \{\chi_{d_N}, \chi_{d_N+1}, \dots, \chi_{r-1}, \chi_r, \chi'_r\} & \text{if } m \text{ even and } \chi_r, \chi'_r \notin \mathcal{S} \end{cases} \\ \mathcal{C}_{\chi_r} &= \{\chi_r\} && \text{if } \chi_r \in \mathcal{S} \\ \mathcal{C}_{\chi'_r} &= \{\chi'_r\} && \text{if } \chi'_r \in \mathcal{S} \\ \mathcal{C}_\epsilon &= \{\epsilon\} \end{aligned}$$

for a suitable sequence of integers $f_1 \geq f_2 \geq \cdots \geq f_N$. (It is possible that $f_k = f_{k+1}$, in which case the second part of \mathcal{C}_k is empty.)

Moreover, except in the case where m is even and \mathcal{S} contains exactly one of χ_r and χ'_r , we actually have $f_k = \lfloor \frac{m-1}{2} \rfloor$ for all k , so $I_2(m)$ admits a unique Springer correspondence whose set of Springer representations is \mathcal{S} .

On the other hand, if m is even and \mathcal{S} contains χ'_r but not χ_r , then for any integers $r = f_1 \geq \cdots \geq f_N \geq d_N$ such that $f_k - f_{k+1} \leq d_{k+1} - d_k$ for each k , there is a Springer correspondence as above.

We introduce the following notation: if $\text{supp } \chi \geq \mathcal{C}$ and $\text{supp } \chi' \geq \mathcal{C}$, then let

$$(7) \quad Y_{\chi, \chi'}^{\mathcal{C}} = \gamma^{-1} \left(\omega_{\chi, \chi'} - \sum_{\mathcal{C}' < \mathcal{C}} P_{\chi, \mathcal{C}'} \Lambda_{\mathcal{C}'} P_{\chi', \mathcal{C}'}^t \right) \quad \text{where} \quad \gamma = q^m - 1$$

It is then clear from (4) that

$$(8) \quad \Lambda_{\mathcal{C}} = q^{-2ac} \gamma (Y_{\chi, \chi'}^{\mathcal{C}})_{\chi, \chi' \in \mathcal{C}}.$$

Below, we will give formulas for $Y^{\mathcal{C}}$ and P . It is then immediate to compute Λ .

Henceforth, for the sake of brevity of notation, we will generally write “ i ” instead of “ χ_i ,” as well as “ r ” for “ χ'_r .” We also write $Y^{(k)}$ for $Y^{\mathcal{C}_k}$.

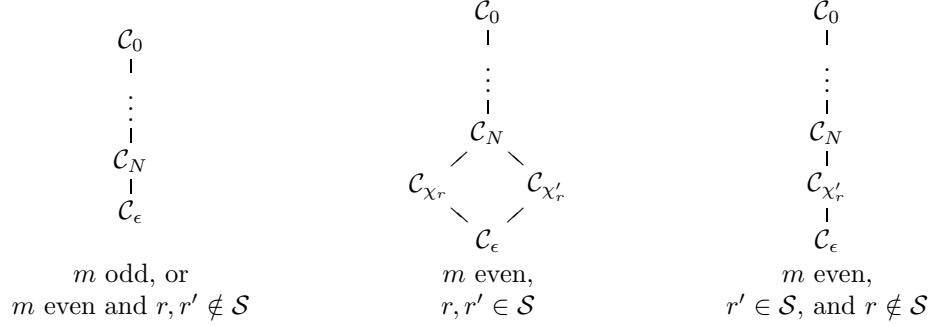


TABLE 1. Closure order of unipotent classes in dihedral groups

Let

$$\iota = \begin{cases} 1 & \text{if } m \text{ is odd, or if } m \text{ is even and } r, r' \notin \mathcal{S}, \\ 0 & \text{if } m \text{ is even, } r' \in \mathcal{S}, \text{ and } r \notin \mathcal{S}, \\ -1 & \text{if } m \text{ is even and } r, r' \in \mathcal{S}. \end{cases}$$

In Section 5, we will establish the following formulas for $Y^{(k)}$ by induction on k :

$$Y_{ij}^{(N)} = q^{m-|i-j|} + \iota q^{i+j} \quad Y_{ir'}^{(N)} = \begin{cases} q^{i+r} & \text{if } i < r, \\ 0 & \text{if } i = r \end{cases}$$

$$Y_{ij}^{(k)} = \begin{cases} q^{m-|i-j|} - q^{m+i+j-2d_{k+1}} & \text{if } i, j < d_{k+1}, \\ 0 & \text{if } i < d_{k+1} \leq f_{k+1} < j \\ & \text{or } j < d_{k+1} \leq f_{k+1} < i, \\ q^{m-|i-j|} - q^{m-i-j+2f_{k+1}} & \text{if } i, j > f_{k+1}. \end{cases}$$

We will simultaneously show that P is given by

$$P_{ij} = \begin{cases} q^i & \text{if } i < j \text{ and } j \in \mathcal{S} \\ q^{d_k+f_k-i} & \text{if } i > j = f_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases} \quad P_{ir'} = \begin{cases} q^i & \text{if } r' \in \mathcal{S} \text{ and } i < r \\ 0 & \text{otherwise} \end{cases}$$

$$P_{\chi, \epsilon} = R(\chi)$$

(Here, the formula for P_{ij} is only valid under the assumption that $\text{supp } i > \text{supp } j$.)

Now, recall that a variety Z is *rationaly smooth* (see [14, Appendix]) if

$$\mathcal{H}_z^i \text{IC}(Z, \bar{\mathbb{Q}}_l) = \begin{cases} \bar{\mathbb{Q}}_l & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } z \in Z.$$

In the original Lusztig–Shoji algorithm, the entries of P enjoy the following interpretation in terms of intersection cohomology complexes: if χ corresponds to the local system \mathcal{E} on the unipotent class C , and likewise χ' corresponds to \mathcal{E}' on C' , then we have

$$(9) \quad P_{\chi, \chi'} = \sum_{i \geq 0} [\mathcal{E}' : \mathcal{H}^i \text{IC}(\bar{C}, \mathcal{E})] q^{a(C)+i},$$

where $a(C)$ is the dimension of the Springer fiber over a point of C . (See [18, Theorem 24.8], but note that the definition of Ω is slightly different there, resulting in a slightly different formula for P .)

If we now interpret the entries of P for a dihedral group via (9) as describing the geometry of some unknown variety, we obtain the following result.

Theorem 3.3. *With respect to any Springer correspondence, every special piece of $I_2(m)$ is rationally smooth. The full unipotent variety is also rationally smooth.*

Proof. The classes \mathcal{C}_0 and \mathcal{C}_ϵ each constitute a special piece by themselves, so they are obviously rationally smooth. For the “middle” special piece, which contains the special class \mathcal{C}_1 , we see that $P_{1d_k} = q$ for all k , and $P_{1r'} = q$, but $P_{1i} = 0$ if i is not a Springer representation, so this piece is rationally smooth as well.

The second sentence is simply the observation that $P_{0i} = 1$ if i is a Springer representation, and $P_{0i} = 0$ otherwise. \square

Now, suppose a group admits multiple Springer correspondences for a fixed set \mathcal{S} of Springer representations. Since \mathcal{S} is fixed, it is possible to identify unipotent classes in distinct Springer correspondences, and it makes sense to compare the support of a given $\chi \in \text{Irr}(W)$ in various Springer correspondences.

In this situation, a Springer correspondence X is called *maximal* if $\text{supp}_X \chi \geq \text{supp}_{X'} \chi$ for all $\chi \in \text{Irr}(W)$ and all other Springer correspondences X' for \mathcal{S} . There is some evidence (see Remark 3.5 below) that the actual Springer correspondences for algebraic groups satisfy a maximality condition of this type; perhaps it is something that should be added to Definition 2.4. In any case, for the dihedral groups, if we require maximality, then \mathcal{S} determines a unique Springer correspondence in all cases.

Theorem 3.4. *For any set \mathcal{S} of irreducible representations of $I_2(m)$ with $\chi_0, \chi_1, \epsilon \in \mathcal{S}$, there is a (necessarily unique) maximal Springer correspondence for $I_2(m)$ whose set of Springer representations is \mathcal{S} .*

Proof. There is only something to prove in the case that m is even, $r' \in \mathcal{S}$, and $r \notin \mathcal{S}$. We define f_k inductively. Let $f_1 = r$, and then for $k > 1$, let $f_k = \max\{d_N, f_{k-1} - (d_k - d_{k-1})\}$. \square

Remark 3.5. Note that the uniqueness proved in Theorem 3.4 is not of “global nature”, in the sense that it depends on the choice of the set \mathcal{S} (the set of Springer representations in our terminology). In the case of the dihedral group of order 8, that is, the Weyl group of type B_2 , this set will not be the same for the Springer correspondences associated to groups in odd characteristic, characteristic 2 or to disconnected groups (see section 4). Hence we get more than one Springer correspondence for the group B_2 .

4. COMPATIBILITY WITH KNOWN SPRINGER CORRESPONDENCES

4.1. Connected algebraic groups in odd characteristic. In the cases $m = 3, 4, 6$, when $I_2(m)$ is in fact the Weyl group of a connected algebraic group of type A_2, B_2, G_2 respectively, it is easy to verify that the unique maximal Springer correspondence of Theorem 3.4 coincides with the “true” Springer correspondence, as found in, say, [10, Section 13.3]. The Springer correspondences for these three groups are given below.

For algebraic groups of types A_2 and B_2 , both the unipotent classes and the representations of the Weyl group are labelled by partitions (or pairs of partitions). In G_2 , unipotent classes are named by their Bala–Carter labels, and representations have been named using Carter’s notation [10]. To identify representations in these

notations with our χ_i 's, it suffices to compute b_χ for all of them. In types A_2 and B_2 , this can be done with [10, Propositions 11.4.1 and 11.4.2], while for G_2 , the required information is included in the notation (we have $b_{\phi_{d,n}} = n$). In types B_2 and G_2 , one has a choice of which representation to label as χ_r and which as χ'_r ; we have made the choice that agrees with Theorem 3.2. The corresponding sets \mathcal{S} are $\{\chi_0, \chi_1, \epsilon\}$, $\{\chi_0, \chi_1, \chi'_2, \epsilon\}$, $\{\chi_0, \chi_1, \chi_2, \chi'_3, \epsilon\}$, in type A_2 , B_2 , G_2 , respectively.

Type A_2		Type B_2		Type G_2	
class	reps.	class	reps.	class	reps.
[3]	$[3] = \chi_0$	[5]	$([2], \emptyset) = \chi_0$	G_2	$\phi_{1,0} = \chi_0$
$[2, 1]$	$[2, 1] = \chi_1$	$[3, 1^2]$	$([1], [1]) = \chi_1; ([1^2], \emptyset) = \chi_2$	$G_2(a_1)$	$\phi_{2,1} = \chi_1; \phi'_{1,3} = \chi_3$
$[1^3]$	$[1^3] = \epsilon$	$[2^2, 1]$	$(\emptyset, [2]) = \chi'_2$	A_1	$\phi_{2,2} = \chi'_2$
		$[1^5]$	$(\emptyset, [1^2]) = \epsilon$	A_1	$\phi'_{1,3} = \chi'_3$
				1	$\phi_{1,6} = \epsilon$

4.2. Bad characteristics. It follows from [34] that for connected algebraic groups of both types B_2 in characteristic 2 and G_2 in characteristic 3, the partially ordered set of unipotent classes has the form of second diagram in Table 1 of the present paper (the one in which the two classes \mathcal{C}_{χ_r} and $\mathcal{C}_{\chi'_r}$ cannot be compared). Then, the explicit computations of the Springer correspondences in these two cases, provided by [21] and [35], show that they coincide with those given by Theorem 3.2. The same references show that the Springer correspondence for G_2 in characteristic 2 coincides with the one in good characteristic. Hence the Springer correspondences associated to reductive groups of rank 2 in bad characteristic are recovered by our theorem.

4.3. Non-connected algebraic groups. We will consider below two Springer correspondences which occur “in nature,” associated to the group B_2 and to two types of disconnected groups.

We assume here that the characteristic equals 2. Let $G(5)$ be the group defined as the extension of $GL(5)$ by the order 2 automorphism of the diagram. There are 5 unipotent classes of $G(5)$ which are not contained in $GL(5)$, and the partial order is here of the kind of the second diagram of Table 1 of section 3 of the present paper. Then Table 2 on page 314 of [26] shows that the Springer correspondence for this group coincides also with the one given by Theorem 3.2. We have $\mathcal{S} = \{\chi_0, \chi_1, \chi_2, \chi'_2, \epsilon\}$.

Let GO_6 be the general orthogonal group over an algebraically closed field of characteristic 2, the extension of SO_6 by a non-trivial graph automorphism of order 2. There are 4 unipotent classes which are not contained in SO_6 , the corresponding partial order between them is of the form of the third diagram of Table 1 of section 3 of the present paper, and Table 4 on page 318 of [26] shows that the restricted Springer correspondence for this group coincides also with the one deduced from Theorem 3.2. We have $\mathcal{S} = \{\chi_0, \chi_1, \chi'_2, \epsilon\}$.

The Springer correspondences for the two above groups are given below, where unipotent classes are labelled as in [26].

Group $G(5)$		Group GO_6	
class	reps.	class	reps.
[5]	$([2], \emptyset) = \chi_0$	[6]	$([2], \emptyset) = \chi_0$
$[3, 1^2]$	$([1], [1]) = \chi_1$	$[4, 1^2]$	$([1], [1]) = \chi_1; ([1^2], \emptyset) = \chi_2$
$[3, 1^3_0]$	$([1^2], \emptyset) = \chi_2$	$[2^3]$	$(\emptyset, [2]) = \chi'_2$
$[2^2, 1]$	$(\emptyset, [2]) = \chi'_2$	$[2, 1^4]$	$(\emptyset, [1^2]) = \epsilon$
$[1^5]$	$(\emptyset, [1^2]) = \epsilon$		

5. PROOF OF THEOREM 3.2

The proof of Theorem 3.2 is quite straightforward: we simply attempt to carry out the Lusztig–Shoji algorithm. If the attempt succeeds, and if the solution satisfies the conditions given in Section 1, then we will have produced a Springer correspondence. In fact, we will find that if the unipotent classes are not as described in the theorem, then the attempt to calculate $P_{\chi, \mathcal{C}}$ fails (there is no solution that is a matrix with entries which are polynomials with nonnegative integer coefficients).

We define

$$f_k = \max\{i \mid 1 \leq i \leq m/2 \text{ and } \text{supp } i \geq \mathcal{C}_k\}.$$

In the case that m is even, $r' \in \mathcal{S}$, and $r \notin \mathcal{S}$, the theorem states that the inequality $f_k - f_{k+1} \leq d_{k+1} - d_k$ must hold. We actually prove below that, irrespective of whether this inequality holds or not, there is a system of Green functions in which the entries of P are Laurent polynomials with nonnegative integer coefficients, and the entries of Λ are polynomials with integer coefficients.

The formula for P given in the previous section holds in this more general setting. The restriction $f_k - f_{k+1} \leq d_{k+1} - d_k$ is an immediate consequence of requiring $P_{\chi, \chi'}$ to be a polynomial divisible by $q^{a_{\text{supp } \chi}}$.

5.1. Proof outline. We will study the unipotent classes in increasing order, starting with the trivial class. For each unipotent class \mathcal{C} , we carry out the following four steps:

5.1.1. *Compute $Y_{\chi, \chi'}^{\mathcal{C}}$, using known formulas for lower classes.* We assume that we are in the case when there is exactly one highest class, say \mathcal{D} , below \mathcal{C} . From the definition of $Y_{\chi, \chi'}^{\mathcal{C}}$, we have

$$\begin{aligned} Y_{\chi, \chi'}^{\mathcal{C}} &= \gamma^{-1} \left(\omega_{\chi, \chi'} - \sum_{\mathcal{C}' < \mathcal{C}} P_{\chi, \mathcal{C}'} \Lambda_{\mathcal{C}'} P_{\chi', \mathcal{C}'}^t \right) \\ &= Y_{\chi, \chi'}^{\mathcal{D}} - \gamma^{-1} P_{\chi, \mathcal{D}} \Lambda_{\mathcal{D}} P_{\chi', \mathcal{D}}^t \end{aligned}$$

In most cases, the matrices $P_{\chi, \mathcal{D}}$ and $P_{\chi', \mathcal{D}}$ each have a single nonzero entry (see 5.1.2), so the above equation reduces to

$$(10) \quad Y_{\chi, \chi'}^{\mathcal{C}} = Y_{\chi, \chi'}^{\mathcal{D}} - q^{-2a_{\mathcal{D}}} P_{\chi, S(\chi)} Y_{S(\chi), S(\chi')}^{\mathcal{D}} P_{\chi', S(\chi')}$$

for suitable characters $S(\chi), S(\chi') \in \mathcal{D}$. Once we obtain this formula, the formula for $\Lambda_{\mathcal{C}}$ follows immediately.

5.1.2. *Set up equations for finding $P_{\chi, \mathcal{C}}$.* From (5) and (8), we have

$$P_{\chi, \mathcal{C}} Y_{\mathcal{C}, \mathcal{C}}^{\mathcal{C}} = q^{ac} Y_{\chi, \mathcal{C}}^{\mathcal{C}}$$

Selecting one entry from both sides of this equation, we have

$$(11) \quad \sum_{\chi'' \in \mathcal{C}} P_{\chi, \chi''} Y_{\chi'', \chi'}^{\mathcal{C}} = q^{ac} Y_{\chi, \chi'}^{\mathcal{C}}$$

The details of the argument from here on vary, but the main ideas are as follows:

- Since $P_{\chi, \chi''}$ is a polynomial with nonnegative coefficients, $P_{\chi, \chi''}|_{q=1}$ is a nonnegative integer. Moreover, this integer is 0 if and only if $P_{\chi, \chi''}$ is the zero polynomial.

- Evaluating the entire equation (11) at $q = 1$, with judicious choices of χ' , may allow one to deduce strong constraints on the χ'' with $P_{\chi, \chi''} \neq 0$.
- If $P_{\chi, \chi''}|_{q=1} = 1$, then $P_{\chi, \chi''}$ is a power of q .

Typically, one shows that there is a unique $\chi'' \in \mathcal{C}$ such that $P_{\chi, \chi''} \neq 0$. Let $S(\chi)$ denote this χ'' . Next, one shows that $P_{\chi, S(\chi)}$ is a power of q , say q^b . By reconsidering (11), one may find an equation relating b to $S(\chi)$.

5.1.3. Determine the members of \mathcal{C} . To prove that $\chi \in \mathcal{C}$, we assume that in fact $\text{supp } \chi > \mathcal{C}$, and we try to use the work of the previous step to compute $P_{\chi, \mathcal{C}}$. (We will have already explicitly determined the members of all lower classes, so we know that $\text{supp } \chi \not\leq \mathcal{C}$.) This calculation leads to a contradiction, from which we deduce that $\chi \in \mathcal{C}$.

5.1.4. Give a formula for $P_{\chi, \mathcal{C}}$. Knowing the members of \mathcal{C} often enables us to obtain further conditions on those χ that actually have $\text{supp } \chi > \mathcal{C}$. Using these conditions, one can determine the values of b and $S(\chi)$ (as defined in 5.1.2), and hence obtain a formula for $P_{\chi, \mathcal{C}}$.

5.2. The trivial unipotent class. For the smallest unipotent class, we cannot use (10) to compute $Y^{\mathcal{C}_\epsilon}$. Rather, we must use (4) and (7) directly:

$$\Lambda_{\epsilon, \epsilon} = q^{-2m}(\omega_{\epsilon, \epsilon}) = 1 \quad \text{and} \quad Y_{\chi, \chi'}^{\mathcal{C}_\epsilon} = \gamma^{-1} \omega_{\chi, \chi'}.$$

Finally, from (5),

$$P_{\chi, \epsilon} = q^{-m}(\omega_{\chi, \epsilon}) \cdot 1 = q^{-m} q^m R(\chi \otimes \epsilon \otimes \epsilon) = R(\chi).$$

5.3. The classes with Springer representation r or r' . We will carry out the appropriate calculations for $\mathcal{C}_{\chi'_r}$. (Recall that we have assumed that if exactly one of χ'_r and χ_r is in \mathcal{S} , then in fact $\chi'_r \in \mathcal{S}$.) If $\chi_r \in \mathcal{S}$ as well, the calculations for \mathcal{C}_{χ_r} are identical.

From a similar argument as in (10), we have

$$Y_{\chi, \chi'}^{\mathcal{C}_{\chi'_r}} = \gamma^{-1} \omega_{\chi, \chi'} - q^{-2m} R(\chi) \gamma^{-1} \omega_{\epsilon, \epsilon} R(\chi') = (\omega_{\chi, \chi'} - R(\chi) R(\chi')) / \gamma.$$

In the following table, we carry out this calculation in all cases. Throughout, we assume that $1 \leq i, j < m/2$.

$Y_{00}^{\mathcal{C}_{\chi'_r}} = (q^{2m} - q^0 q^0) / \gamma = q^m + 1$	$Y_{ij}^{\mathcal{C}_{\chi'_r}} = (q^{m+ i-j } + q^{m+i+j} + q^{2m-i-j} + q^{2m- i-j } - (q^i + q^{m-i})(q^j + q^{m-j})) / \gamma$
$Y_{0j}^{\mathcal{C}_{\chi'_r}} = (q^{m+j} + q^{2m-j} - q^0(q^j + q^{m-j})) / \gamma = q^{m-j} + q^j$	$= (q^{2m- i-j } + q^{m+ i-j } - q^{m-i+j} - q^{m+i-j} + q^{m+i+j} - q^{i+j}) / \gamma = q^{m- i-j } + q^{i+j}$
$Y_{0r}^{\mathcal{C}_{\chi'_r}} = (q^{3r} - q^0 q^r) / \gamma = q^r$	$Y_{rr}^{\mathcal{C}_{\chi'_r}} = (q^{2m} - q^r q^r) / \gamma = q^{2r}$
$Y_{ir}^{\mathcal{C}_{\chi'_r}} = (q^{3r-i} + q^{3r+i} - (q^i + q^{m-i}) q^r) / \gamma = q^{r+i}$	$Y_{r,r'}^{\mathcal{C}_{\chi'_r}} = (q^m - q^r q^r) / \gamma = 0$

By a fortunate coincidence, we can combine many of these formulas. Below, we permit $0 \leq i, j \leq m/2$:

$$Y_{ij}^{\mathcal{C}_{\chi'_r}} = \begin{cases} q^{m-|i-j|} + q^{i+j} & \text{if } i, j < m/2, \\ q^{i+j} & \text{if } i = r \text{ or } j = r. \end{cases}$$

Next, (11) gives

$$P_{ir'} \cdot q^{2r} = q^r Y_{ir'}^{\mathcal{C}_{\chi'_r}} \quad \text{if } i < r,$$

from which it follows that $P_{ir'} = q^i$.

5.4. General arguments for the class \mathcal{C}_N .

5.4.1. *Formula for $Y^{(N)}$.* From (10), we have

$$\begin{aligned} Y_{ij}^{(N)} &= Y_{ij}^{\mathcal{C}_{x'_r}} - (1 - \iota)q^{-2r}P_{ir'}Y_{r'r'}^{\mathcal{C}_{x'_r}}P_{jr'} \\ &= \begin{cases} q^{m-|i-j|} + q^{i+j} - (1 - \iota)q^{-2r}q^i(q^{2r})q^j & \text{if } i, j < r, \\ q^{i+j} - (1 - \iota) \cdot 0 & \text{if } i = r \text{ or } j = r \end{cases} \end{aligned}$$

We can now set up the various versions of (11) that we will require. Let

$$\delta = \begin{cases} 0 & \text{if } r \notin \mathcal{C}_N, \\ 1 & \text{if } r \in \mathcal{C}_N, \end{cases} \quad \delta' = \begin{cases} 0 & \text{if } r' \notin \mathcal{C}_N, \\ 1 & \text{if } r' \in \mathcal{C}_N. \end{cases}$$

First, suppose $i < m/2$. For all $j \in \mathcal{C}_N$, $j < r$, we have

$$(12) \quad \sum_{\substack{l \in \mathcal{C}_N \\ l < r}} P_{il}(q^{m-|l-j|} + \iota q^{l+j}) + \delta P_{ir}q^{j+r} + \delta' P_{ir'}q^{j+r} = q^{d_N}(q^{m-|i-j|} + \iota q^{i+j}).$$

If, in fact, $\delta = 1$, so that $r \in \mathcal{C}_N$, then we also have

$$(13) \quad \sum_{\substack{l \in \mathcal{C}_N \\ l < r}} P_{il}q^{l+r} + P_{ir}q^{2r} + \delta' P_{ir'} \cdot 0 = q^{d_N}q^{i+r}.$$

An analogue of this holds if $\delta' = 1$.

On the other hand, if $\text{supp } r > \mathcal{C}_N$, then we have

$$(14) \quad \sum_{\substack{l \in \mathcal{C}_N \\ l < r}} P_{rl}(q^{m-|l-j|} + \iota q^{l+j}) + \delta' P_{rr'}q^{j+r} = q^{d_N}q^{j+r},$$

and, additionally, if $\delta' = 1$,

$$(15) \quad \sum_{\substack{l \in \mathcal{C}_N \\ l < r}} P_{rl}q^{l+r} + P_{rr'}q^{2r} = q^{d_N} \cdot 0.$$

5.4.2. *Calculation of P_{i, \mathcal{C}_N} .* Let us assume now that we have shown that the left-hand side of (12) reduces to a single nonzero term, indexed by some $l \in \mathcal{C}_N$, and that, moreover, the coefficient P_{il} is a power of q , say $P_{il} = q^b$. (This will require slightly different arguments depending on ι .) So (12) reduces to

$$(16) \quad \begin{aligned} q^b(q^{m-|l-j|} + \iota q^{l+j}) &= q^{d_N}(q^{m-|i-j|} + \iota q^{i+j}) \\ q^{b+m-|l-j|} + \iota q^{b+l+j} &= q^{d_N+m-|i-j|} + \iota q^{d_N+i+j} \end{aligned}$$

for all $j \in \mathcal{C}_N$, $j < m/2$. In particular, for $j = d_N$, the preceding equation becomes

$$(17) \quad q^{b+m-l+d_N} + \iota q^{b+l+d_N} = q^{m-|i-d_N|+d_N} + \iota q^{i+2d_N}$$

If $\iota = 1$, we must decide which term on the left corresponds to each term on the right. We do this by comparing exponents. Since $l < m/2$, we evidently have $b + m - l + d_N > b + l + d_N$. Similarly,

$$m - |i - d_N| + d_N = \begin{cases} m + i & (\text{if } i < d_N) \\ m - i + 2d_N & (\text{if } i > d_N) \end{cases} \quad \begin{matrix} > i + 2d_N & \text{since } d_N < m/2, \\ > i + 2d_N & \text{since } i < m/2. \end{matrix}$$

Therefore, (17) implies that

$$q^{b+m-l+d_N} = q^{m-|i-d_N|+d_N} \quad \text{and, if } \iota = 1, \quad q^{b+l+d_N} = q^{i+2d_N},$$

and hence

$$(18) \quad b = l - |i - d_N| \quad \text{and, if } \iota = 1, \quad b = i + d_N - l.$$

5.5. The class \mathcal{C}_N when $\iota = 1$. We will begin by showing that if m is even, then $r, r' \in \mathcal{C}_N$. Suppose, for instance, that $\text{supp } r > \mathcal{C}_N$ instead. Then, (14) evaluated at $q = 1$ can hold only if $P_{rl} = 0$ for all $l < r$, and $\delta' P_{rr'}|_{q=1} = 1$. In particular, it is necessarily the case that $\delta' = 1$. But now these values for P_{rl} and $P_{rr'}$ violate (15). So it must be that $r \in \mathcal{C}_N$. The same argument, with the roles of r and r' reversed, shows that $r' \in \mathcal{C}_N$.

Now, suppose that $i < m/2$. If m is even, then comparing (13) with the analogous equation in which r and r' are exchanged (recall that $\delta = \delta' = 1$) shows that $P_{ir} = P_{ir'}$. Suppose they are both nonzero. Then, evaluating (12) at $q = 1$ shows that in fact $P_{ir}|_{q=1} = P_{ir'}|_{q=1} = 1$ (since we are in the case $\iota = 1$), and, furthermore, that $P_{il} = 0$ for all $l < r$. P_{ir} and $P_{ir'}$ must be powers of q ; say $P_{ir} = P_{ir'} = q^b$. Now (12) reduces to

$$2q^b q^{j+r} = q^{d_N+m-|i-j|} + q^{d_N+i+j}.$$

In order for there to be a b satisfying this equation for all $j \in \mathcal{C}_N$, $j < r$, it must be that $d_N + m - |i - j| = d_N + i + j$ for all such j . Therefore,

$$m = i + j + |i - j| = 2 \max\{i, j\},$$

in contradiction to the fact that $i, j < m/2$. We conclude that $P_{ir} = P_{ir'} = 0$.

We now drop the assumption that m is even, and we return to (12) with the knowledge that the last two terms of the left-hand side vanish in all cases. Again considering that equation at $q = 1$, we see that there is a unique $l \in \mathcal{C}_N$, $l < m/2$, such that P_{il} is nonzero. For that l , we know that $P_{il}|_{q=1} = 1$, so P_{il} must be a power of q , say q^b . We are therefore in the situation of Section 5.4.2, and (18) holds.

5.5.1. Members of \mathcal{C}_N . We have already shown at the beginning of the section that $r, r' \in \mathcal{C}_N$. Now, suppose that $d_N < i < m/2$. If $i \notin \mathcal{C}_N$, then we can calculate P_{il} as above, and the formulas in (18) say

$$b = l - i + d_N = i + d_N - l.$$

But this implies that $i = l$, which is absurd. Therefore, if $d_N < i < m/2$, it must be that $i \in \mathcal{C}_N$. We conclude that

$$\mathcal{C}_N = \begin{cases} \{d_N, d_N + 1, \dots, (m-1)/2\} & \text{if } m \text{ is odd,} \\ \{d_N, d_N + 1, \dots, r-1, r, r'\} & \text{if } m \text{ is even.} \end{cases}$$

In particular, $f_N = \lfloor \frac{m-1}{2} \rfloor$ in all cases.

5.5.2. *Calculation of P_{i, \mathcal{C}_N} .* It remains to consider the case $i < d_N$. We must determine the unique l such that $P_{il} \neq 0$, and then we must give a formula for that P_{il} . The formulas (18) now say

$$b = l - d_N + i = i + d_N - l.$$

These imply that $l = d_N$, and hence that $b = i$. Thus, if $i < d_N$ and $j \in \mathcal{C}_N$, we have

$$P_{ij} = \begin{cases} q^i & \text{if } j = d_N, \\ 0 & \text{otherwise.} \end{cases}$$

5.6. **The class \mathcal{C}_N when $\iota = 0$.** In this case, we obviously have $\delta' = 0$ (as r' is the Springer representation of a smaller class). Moreover, since $q^{j+r} = q^{m-|r-j|}$, the equation (12) can be rewritten in the more concise form

$$\sum_{l \in \mathcal{C}_N} P_{il} q^{m-|l-j|} = q^{d_N} q^{m-|i-j|}.$$

Evaluating this at $q = 1$ shows that there is a unique $l \in \mathcal{C}_N$ such that $P_{il} \neq 0$, and that for that l , P_{il} is a power of q . As before, the arguments of Section 5.4.2 hold. There is now only one term on each side of (16), and we conclude that $b + m - |l - j| = d_N + m - |i - j|$, or

$$(19) \quad b = d_N - |i - j| + |l - j| \quad \text{for all } j \in \mathcal{C}_N$$

5.6.1. *Members of \mathcal{C}_N .* Suppose $d_N < i < f_N$. If $\text{supp } i > \mathcal{C}_N$, then we could use the above formula to compute P_{i, \mathcal{C}_N} . Putting $j = d_N$ and $j = f_N$ respectively in (19), we find that

$$\begin{aligned} b &= d_N - (i - d_N) + (l - d_N) = d_N - i + l \\ b &= d_N - (f_N - i) + (f_N - l) = d_N + i - l \end{aligned}$$

This equations together imply that $i = l$, which is absurd. Therefore,

$$\mathcal{C}_N = \{d_N, d_N + 1, \dots, f_N\}.$$

5.6.2. *Calculation of P_{i, \mathcal{C}_N} .* We now know that either $i < d_N$ or $i > f_N$. Suppose $i < d_N$. As above, putting $j = f_N$ into (19) shows that $b = d_N + i - l$. On the other hand, the first formula in (18) simplifies to $b = l - d_N + i$. These two formulas together imply that $l = d_N$ and $b = i$.

Now, suppose that $i > f_N$. This time, putting $j = f_N$ into (19) gives

$$b = d_N - (i - f_N) + (f_N - l) = d_N - i - l + 2f_N.$$

On the other hand, (18) now says that $b = l - i + d_N$. These two together imply that $l = f_N$, and that $b = d_N + f_N - i$. We conclude that if $i \notin \mathcal{C}_N$ and $j \in \mathcal{C}_N$, then

$$P_{ij} = \begin{cases} q^i & \text{if } i < d_N \text{ and } j = d_N, \\ q^{d_N + f_N - i} & \text{if } i > f_N \text{ and } j = f_N, \\ 0 & \text{otherwise.} \end{cases}$$

5.7. The class \mathcal{C}_N when $\iota = -1$. The calculations required in this case are nearly identical to those to be done in Section 5.8. Indeed, by introducing the notation $d_{N+1} = f_{N+1} = m$, we see that (12) is the same as the first equation in (20) below, with $k = N$. Some of the cases considered below are inapplicable here, since the inequalities $i > f_{N+1}$ and $j > f_{N+1}$ cannot occur. To determine \mathcal{C}_N and calculate P_{i,\mathcal{C}_N} , however, we simply quote the appropriate portions of the results from the following section. We have

$$\mathcal{C}_N = \{d_N, d_N + 1, \dots, r - 1\}$$

and, if $i < d_N$ and $j \in \mathcal{C}_N$,

$$P_{ij} = \begin{cases} q^i & \text{if } j = d_N, \\ 0 & \text{otherwise.} \end{cases}$$

5.8. Larger unipotent classes.

5.8.1. Formula for $Y^{(k)}$. From (10), we have

$$Y_{\chi,\chi'}^{(k)} = Y_{\chi,\chi'}^{(k+1)} - q^{-2d_{k+1}} P_{\chi,S(\chi)} Y_{S(\chi),S(\chi')}^{(k+1)} P_{\chi',S(\chi')}.$$

We will first treat the case $k = N - 1$. If $i, j < d_N$, we have

$$\begin{aligned} Y_{ij}^{(N-1)} &= q^{m-|i-j|} + \iota q^{i+j} - q^{-2d_N} q^i (q^m + \iota q^{2d_N}) q^j \\ &= q^{m-|i-j|} - q^{m+i+j-2d_N}. \end{aligned}$$

Next, if $i < d_N$ and $f_N < j$, then $\iota = 0$, and we have

$$Y_{ij}^{(N-1)} = q^{m-(j-i)} - q^{-2d_N} q^i (q^{m-(f_N-d_N)}) q^{d_N+f_N-j} = 0.$$

Finally, if $i, j > f_N$, then again $\iota = 0$, and

$$\begin{aligned} Y_{ij}^{(N-1)} &= q^{m-|i-j|} - q^{-2d_N} q^{d_N+f_N-i} (q^m) q^{d_N+f_N-j} \\ &= q^{m-|i-j|} - q^{m-i-j+2f_N}. \end{aligned}$$

Next, we carry out analogous calculations for $Y^{(k)}$, assuming that the formulas for $Y^{(k+1)}$ and the various P_{ij} for $j \in \mathcal{C}_{k+1}$ are known. If $i, j < d_{k+1}$, then

$$\begin{aligned} Y_{ij}^{(k)} &= q^{m-|i-j|} - q^{m+i+j-2d_{k+2}} - q^{-2d_{k+1}} q^i (q^m - q^{m+2d_{k+1}-2d_{k+2}}) q^j \\ &= q^{m-|i-j|} - q^{m+i+j-2d_{k+1}}. \end{aligned}$$

Next, if $i < d_{k+1}$ and $j > f_{k+1}$, then

$$Y_{ij}^{(k)} = 0 - q^{-2d_{k+1}} q^i \cdot 0 = 0.$$

(It should be noted that the 0 in the second term may arise in two different ways: if $f_{k+1} \in \mathcal{C}_{k+1}$, then $Y_{d_{k+1},f_{k+1}}^{(k+1)} = 0$, but on the other hand, if $f_{k+1} \notin \mathcal{C}_{k+1}$, then $P_{j,\mathcal{C}_{k+1}} = 0$.) Finally, if $i, j > f_{k+1}$, then if $f_{k+1} \notin \mathcal{C}_{k+1}$ (which implies $f_{k+1} = f_{k+2}$), we have

$$Y_{ij}^{(k)} = q^{m-|i-j|} - q^{m-i-j+2f_{k+2}} - q^{-2d_{k+1}} \cdot 0 = q^{m-|i-j|} - q^{m-i-j+2f_{k+1}}.$$

On the other hand, if $f_{k+1} \in \mathcal{C}_{k+1}$, then

$$\begin{aligned} Y_{ij}^{(k)} &= q^{m-|i-j|} - q^{m-i-j+2f_{k+2}} \\ &\quad - q^{-2d_{k+1}} q^{d_{k+1}+f_{k+1}-i} (q^m - q^{m-2f_{k+1}+2f_{k+2}}) q^{d_{k+1}+f_{k+1}-j} \\ &= q^{m-|i-j|} - q^{m-i-j+2f_{k+1}} \end{aligned}$$

5.8.2. *Calculation of P_{i,\mathcal{C}_k} .* Suppose $\text{supp } i > \mathcal{C}_k$ and $j \in \mathcal{C}_k$. From the description of classes below \mathcal{C}_k , we know that $i, j \notin \{d_{k+1}, d_{k+1} + 1, \dots, f_{k+1}\}$. There are therefore four versions of (11) to consider, depending on whether $i < d_{k+1}$ or $i > f_{k+1}$, and whether $j < d_{k+1}$ or $j > f_{k+1}$.

Note that if $j < d_{k+1}$, then all terms on the left-hand side of (11) with $l > f_{k+1}$ vanish, since $Y_{jl}^{(k)} = 0$ for those terms. Likewise, if $j > f_{k+1}$, then all terms with $l < d_{k+1}$ vanish.

$$\begin{aligned} \sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il} (q^{m-|l-j|} - q^{m+l+j-2d_{k+1}}) &= q^{d_k} (q^{m-|i-j|} - q^{m+i+j-2d_{k+1}}) \text{ if } i, j < d_{k+1}, \\ \sum_{\substack{l \in \mathcal{C}_k \\ l > f_{k+1}}} P_{il} (q^{m-|l-j|} - q^{m-l-j+2f_{k+1}}) &= 0 \quad \text{if } i < d_{k+1}, j > f_{k+1}, \\ \sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il} (q^{m-|l-j|} - q^{m+l+j-2d_{k+1}}) &= 0 \quad \text{if } i > f_{k+1}, j < d_{k+1}, \\ \sum_{\substack{l \in \mathcal{C}_k \\ l > f_{k+1}}} P_{il} (q^{m-|l-j|} - q^{m-l-j+2f_{k+1}}) &= q^{d_k} (q^{m-|i-j|} - q^{m-i-j+2f_{k+1}}) \text{ if } i, j > f_{k+1}. \end{aligned} \tag{20}$$

Now, evaluating these equations at $q = 1$ is useless—both sides vanish—but if we differentiate with respect to q first, we obtain useful information. Starting from the first equation in (20), we obtain

$$\begin{aligned} &\sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} \frac{dP_{il}}{dq} (q^{m-|l-j|} - q^{m+l+j-2d_{k+1}}) \\ &\quad + \sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il} ((m-|l-j|)q^{m-|l-j|-1} - (m+l+j-2d_{k+1})q^{m+l+j-2d_{k+1}-1}) \\ &= (d_k + m - |i-j|)q^{d_k+m-|i-j|-1} \\ &\quad - (d_k + m + i + j - 2d_{k+1})q^{d_k+m+i+j-2d_{k+1}-1}. \end{aligned}$$

Evaluating at $q = 1$, we obtain

$$\sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il}|_{q=1} (2d_{k+1} - l - j - |l-j|) = 2d_{k+1} - i - j - |i-j|.$$

Now, $l + j + |l-j| = 2 \max\{l, j\}$, and likewise for the right-hand side, so we have

$$(21) \quad \sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il}|_{q=1} (d_{k+1} - \max\{l, j\}) = d_{k+1} - \max\{i, j\} \quad \text{if } i, j < d_{k+1}.$$

Analogous calculations starting from the other equations in (20) yield

$$(22) \quad \sum_{\substack{l \in \mathcal{C}_k \\ l > f_{k+1}}} P_{il}|_{q=1} (\min\{l, j\} - f_{k+1}) = 0 \quad \text{if } i < d_{k+1}, j > f_{k+1},$$

$$(23) \quad \sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il}|_{q=1} (d_{k+1} - \max\{l, j\}) = 0 \quad \text{if } i > f_{k+1}, j < d_{k+1},$$

$$(24) \quad \sum_{\substack{l \in \mathcal{C}_k \\ l > f_{k+1}}} P_{il}|_{q=1} (\min\{l, j\} - f_{k+1}) = \min\{i, j\} - f_{k+1} \quad \text{if } i, j > f_{k+1}.$$

Note that $\min\{l, j\} - f_{k+1} > 0$ for all $l > f_{k+1}$. Thus, (22) implies that if $i < d_{k+1}$, then $P_{il}|_{q=1} = 0$ for all $l > f_{k+1}$, and hence $P_{il} = 0$ for all $l > f_{k+1}$. Similarly, (23) implies that if $i > f_{k+1}$, then $P_{il} = 0$ for all $l < d_{k+1}$.

Now, let $g = \max\{l \in \mathcal{C}_k \mid l < d_{k+1}\}$, and let $h = \min\{l \in \mathcal{C}_k \mid l > f_{k+1}\}$. (Of course, h might not exist, if \mathcal{C}_k contains no members larger than f_{k+1} .) Putting $j = g$ into (21) and $j = h$ into (24), we find that

$$(25) \quad \sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il}|_{q=1} (d_{k+1} - g) = d_{k+1} - \max\{i, g\} \quad \text{if } i, j < d_{k+1},$$

$$(26) \quad \sum_{\substack{l \in \mathcal{C}_k \\ l > f_{k+1}}} P_{il}|_{q=1} (h - f_{k+1}) = \min\{i, h\} - f_{k+1} \quad \text{if } i, j > f_{k+1}.$$

Note that if $g < i < d_{k+1}$, then no positive integers $P_{il}|_{q=1}$ satisfy (25), since $d_{k+1} - g > d_{k+1} - i > 0$. Thus, if $i < d_{k+1}$, then it is necessarily the case that $i < g$ as well. Now, (25) says that

$$\sum_{\substack{l \in \mathcal{C}_k \\ l < d_{k+1}}} P_{il}|_{q=1} (d_{k+1} - g) = d_{k+1} - g,$$

so as usual, there is a unique l such that P_{il} is nonzero, and for that l , P_{il} is a power of q , say q^b . The first equation in (20) now reduces to

$$q^b (q^{m-|l-j|} - q^{m+l+j-2d_{k+1}}) = q^{d_k} (q^{m-|i-j|} - q^{m+i+j-2d_{k+1}}).$$

By matching exponents of corresponding terms, we see that $b + m - |l - j| = d_k + m - |i - j|$, and $b + m + l + j - 2d_{k+1} = d_k + m + i + j - 2d_{k+1}$. Therefore,

$$(27) \quad b = d_k + |l - j| - |i - j| \quad \text{and} \quad b = d_k + i - l.$$

Similarly, if $i > f_{k+1}$, it follows from (26) that $i > h$. One then deduces that there is a unique nonzero P_{il} , and that it is of the form q^b , where

$$(28) \quad b = d_k + |l - j| - |i - j| \quad \text{and} \quad b = d_k - i + l.$$

5.8.3. Members of \mathcal{C}_k . We wish to show that if $d_k < i < d_{k+1}$, or if $f_{k+1} < i < f_k$, then i necessarily belongs to \mathcal{C}_k . As usual, we show this by trying to calculate P_{i, \mathcal{C}_k} and deriving a contradiction.

Suppose that $d_k < i < d_{k+1}$. Putting $j = d_k$ into (27) gives

$$b = d_k + (l - d_k) - (i - d_k) = d_k + l - i \quad \text{and} \quad b = d_k + i - l.$$

Together, these imply that $i = l$, which is absurd. Similarly, if $f_{k+1} < i < f_k$, then we put $j = f_k$ in (28) to find that

$$b = d_k + (f_k - l) - (f_k - i) = d_k - l + i \quad \text{and} \quad b = d_k + i - l,$$

again deducing that $i = l$. Thus, if $d_k < i < d_{k+1}$ or $f_{k+1} < i < f_k$, it cannot be the case that $\text{supp } i > \mathcal{C}_k$. We already know that $\text{supp } i \not\leq \mathcal{C}_k$, so we conclude that

$$\mathcal{C}_k = \{d_k, d_k + 1, \dots, d_{k+1} - 1\} \cup \{f_{k+1} + 1, f_{k+1} + 2, \dots, f_k\}.$$

5.8.4. *Calculation of P_{i, \mathcal{C}_k} .* We now know that either $i < d_k$ or $i > f_k$. In the former case, we put $j = d_k$ into (27) and find that

$$b = d_k + (l - d_k) - (d_k - i) = i + l - d_k \quad \text{and} \quad b = d_k + i - l.$$

Together, these imply that $b = i$ and $l = d_k$. Similarly, if $i > f_k$, putting $j = f_k$ in (28) gives

$$b = d_k + (f_k - l) - (i - f_k) = d_k - l - i + 2f_k \quad \text{and} \quad b = d_k - i + l,$$

which implies that $l = f_k$ and $b = d_k + f_k - i$. We conclude that

$$P_{ij} = \begin{cases} q^i & \text{if } i < d_k \text{ and } j = d_k, \\ q^{d_k + f_k - i} & \text{if } i > f_k \text{ and } j = f_k, \\ 0 & \text{otherwise.} \end{cases}$$

6. A PREFERRED SET OF SPRINGER REPRESENTATIONS

In this section, we will describe the construction of the set of Springer representations that we have obtained in [1].

Recall first that any reflection subgroup of $I_2(m)$ is isomorphic to a group $I_2(d)$ where $d|m$. We set

$$s_i = \begin{bmatrix} 0 & \zeta^i \\ \zeta^{-i} & 0 \end{bmatrix},$$

and for each divisor d of m (including $d = m$), we identify $I_2(d)$ with the subgroup of $\text{GL}(V)$ generated by s_0 and $s_{m/d}$. In the case when m/d is even, we denote by $I'_2(d)$ the subgroup of $I_2(m)$ generated by s_1 and $s_{m/d+1}$. The group $I'_2(d)$ is isomorphic to $I_2(d)$, but is not conjugate to it. Any reflection subgroup of $I_2(m)$ is conjugate to a $I_2(d)$ or to a $I'_2(d)$.

We will denote the irreducible representations of $I_2(d)$ by

$$\chi_0^{(d)}, \chi_1^{(d)}, \dots, \chi_{\lfloor (d-1)/2 \rfloor}^{(d)}; \quad \epsilon^{(d)}; \quad \text{and, if } d \text{ is even,} \quad \chi_{d/2}^{(d)}, \chi_{d/2}^{(d)'},$$

with $\chi_i^{(m)} = \chi_i$, for $0 \leq i \leq \lfloor (m-1)/2 \rfloor$, $\epsilon^{(m)} = \epsilon$, and, if m is even, $\chi_r^{(m)'} = \chi'_r$, where $r = m/2$.

We have $j_{I_2(d)}^{I_2(m)}(\chi_i^d) = j_{I'_2(d)}^{I_2(m)}(\chi_i^d) = \chi_i$, for $i = 0, 1$. On the other hand, we can fix a choice between χ_r and χ'_r in order that the following holds

$$j_{I_2(d)}^{I_2(m)}(\epsilon^{(d)}) = \begin{cases} \chi_d & \text{if } d \neq m/2, \\ \chi'_r & \text{if } d = r = m/2 \end{cases}$$

and

$$j_{I'_2(d)}^{I_2(m)}(\epsilon^{(d)}) = \chi_d.$$

We will define a subset \mathcal{S}_{pf} of $\text{Irr}(I_2(m))$ as follows. If $m = 2$, we put $\mathcal{S}_{\text{pf}} = \text{Irr}(I_2(2))$. If $m > 2$ is odd,

$$\mathcal{S}_{\text{pf}} = \{\chi_0, \chi_1, \epsilon\} \cup \{\chi_d \mid d \text{ divides } m \text{ and is a power of a prime number}\}.$$

If $m > 2$ is even, and $r = m/2$,

$$\mathcal{S}_{\text{pf}} = \{\chi_0, \chi_1, \epsilon\} \cup \{\chi_d \mid d \neq r \text{ divides } m \text{ and is a power of a prime number}\} \cup \{\chi'_r\}.$$

In [1, Def. 8.5], a notion of *pseudoparabolic subgroup* of a finite complex reflection group has been introduced. Then Theorem 8.13 of [1] says that the set \mathcal{S}_{pf} is the set of all the irreducible representations of $I_2(m)$ which are obtained by truncated induction from special representations of pseudoparabolic subgroups.

We write m as $m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where p_1, p_2, \dots, p_k are prime numbers such that $p_1 < p_2 < \cdots < p_k$. The sequence of integers (6) $d_0 < d_1 < \cdots < d_{N-1} < d_N$ with respect to $\mathcal{S} = \mathcal{S}_{\text{pf}}$ satisfies

$$d_{n_1+n_2+\cdots+n_{i-1}+l} = p_1^{n_1} p_2^{n_2} \cdots p_{i-1}^{n_{i-1}} p_i^{l-1},$$

for $1 \leq i \leq k-1$ and $0 \leq l \leq n_i$, with $N = n_1 + \cdots + n_k$ and $d_N = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k-1}$ if m is not a power of 2 and with $N = n-1$ and $d_N = 2^{n-2}$ if $m = 2^n$, $n \geq 2$.

Example 6.1. Assume that $m = 2p$ with p an odd prime number. Here we have $N = 2$, $d_2 = p = r$ and $\mathcal{S}_{\text{pf}} = \{\chi_0, \chi_1, \epsilon\} \cup \{\chi_2\} \cup \{\chi'_p\}$. From Theorems 3.2 and 3.4, we obtain that the maximal Springer correspondence with respect to \mathcal{S}_{pf} has the form $\mathcal{C}_0 = \{\chi_0\}$, $\mathcal{C}_1 = \{\chi_1, \chi_p\}$, $\mathcal{C}_2 = \{\chi_2, \chi_3, \dots, \chi_{p-1}\}$, $\mathcal{C}'_{\chi_p} = \{\chi'_p\}$ and $\mathcal{C}_\epsilon = \{\epsilon\}$. It contains the actual Springer correspondence of $G_2 = I_2(6)$ in good characteristic as a special case (see section 4.1).

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